Lily's "Invisible Fingers" Strategy for Solving Subtraction Fact Problems

This story is a part of the series:

What's Next? **Stories of Teachers Engaging in Collaborative Inquiry Focused on Using Student Thinking to Inform Instructional Decisions**

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Stories of teachers engaging in collaborative inquiry focused on using student thinking to inform instructional decisions

Editors
Robert C. Schoen
Zachary Champagne

Contributing Authors
Amanda M. Tazaz
Charity Bauduin
Claire Riddell
Naomi Iuhasz-Velez
Robert C. Schoen
Tanya Vik Blais
Wendy Bray
Zachary Champagne

Copy Editor
Anne B. Thistle

Layout and Design
Casey Yu

Workshop Leaders
Linda Levi (Coordinator)
Annie Keith
Debbie Gates
Debbie Plowman Junk
Jae Baek
Joan Case
Luz Maldonado
Olof Steinthorsdottir
Susan Gehn
Tanya Vik Blais

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Introduction

In the context of a professional-development session, a group of teachers designed a lesson for a first-grade class with the intent to move students toward greater fluency with subtraction facts. The teachers first interviewed the students to learn how they were solving subtraction-fact problems. On the basis of their observation that the majority of students were using a counting down by ones strategy, a lesson was designed to encourage the class to begin to use derived-fact strategies.

Relevant Florida Mathematics Standards

MAFS.1.OA.3.6 - Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13)

Background Information

Consider reading chapter three in *Children's Mathematics: Cognitively Guided Instruction* (Carpenter et al., 2015). This chapter provides background on the varied ways children solve addition and subtraction problems, including the use of *known and derived fact strategies*. It also expands upon the strategies explained in the ***Analyzing Student Thinking*** section.

Another discussion on fact fluency is contained in the article "Fluency with basic addition" (Kling, 2011), which discusses components of fluency and how they are addressed in curriculum standards for mathematics.

For more information on assessing fact fluency, consider reading "Assessing basic fact fluency" (Kling & Bay-Williams, 2014), which provides a variety of ways to assess students' fluency with basic math facts.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's Mathematics: Cognitively Guided Instruction (2nd ed.)*. Portsmouth, NH: Heinemann.

Kling, G. (2011). Fluency with basic addition. *Teaching Children Mathematics, 18*(2), 80–88.

Kling, G., & Bay-Williams, J. M. (2014). Assessing basic fact fluency. *Teaching Children Mathematics, 20*(8), 488–497.

Analyzing Student Thinking

A group of teachers conducted one-on-one interviews with all of the students in a classroom of first-grade students to assess their fluency with subtraction facts. The interviews involved verbally posing a series of subtraction problems. On each item, after the students provided their answers, the interviewer asked them to explain how they arrived at the answer. The teacher interviewer had the flexibility to skip any problem(s) thought to pose too much of a challenge for any particular student. During the interview, the students had access to markers, paper, and manipulatives (e.g., linking cubes, base-ten blocks, fingers), but they were not required to use them.

Fact Fluency Items

Students were asked to evaluate each one of the following items, starting at the top of the left-hand column and proceeding down the column, and then starting at the top of the right-hand column and proceeding down the column.

4 – 2	15 – 7
10 – 5	13 – 6
6 – 2	12 – 5
8 – 4	17 – 8
12 – 4	12 – 3
14 – 7	16 – 9
11 – 2	18 – 9
12 – 8	15 – 8
14 – 6	13 – 9

As students responded to each item, the teachers considered whether their thinking processes fit the description of each of the following named strategies: *direct modeling, counting down, counting on to, fact recall—derived facts*, and *fact recall—known facts*[1]. The strategies are explained in more detail in the following sections. For brevity, we will use the terms minuend for the number given before the subtraction symbol and subtrahend for the number provided after the subtraction symbol. For example, in the equation 8 – 5 = 3, the minuend is eight, the subtrahend is five, and the difference is three. Because addition is commutative, a single term, *addend*, will be used for the numbers before and after the addition symbol.

Direct Modeling

A student using a *direct modeling* strategy to solve the problem represents each and every quantity in the problem with some sort of object (e.g., manipulatives, fingers, drawings). For example, when evaluating 8 – 4, the student who directly models may create a set of eight counters (or show eight fingers), remove four of them from the set, and then count the remaining counters to determine the difference is four.

Counting Strategies

A student using the *counting down* strategy starts counting at the minuend and counts backwards a number of times corresponding to the subtrahend to determine the difference. Physical objects may represent numbers, but not every quantity is represented physically. (The most common objects are fingers, and they are used to keep track of how many things have been added or subtracted.) For a problem such as 12 – 4, a student using a *counting down* strategy might begin the count at 12 and say, "12, 11, 10, 9" and then pause and say "8." A slightly different variation is to start the count at 11 and say, "11, 10, 9, 8," and know the answer is 8. In the first example, the student is thinking about taking away the 12, then the 11, then the 10, and finally the nine, leaving eight remaining. In the second, the student is taking away one and then starting the count at 11 and counting back four to arrive at eight.

A student using the *counting on to* strategy begins the count at the subtrahend and counts on to the minuend to determine the difference between the two values. For example, when solving 12 – 8, the student would begin the count at eight and say "9, 10, 11, 12" (probably holding up a finger for each of the numbers in that count) and keep track of the number of words that were said to determine the difference of four.

A student using a *counting down to* strategy begins the count at the minuend and counts down to the subtrahend to determine the difference. For example, when evaluating 12 – 8, the student would begin the count at 12, say "11, 10, 9, 8," (usually holding up a finger for each of the numbers in that count), and keep track of the number of words that were said to determine the difference of four. This strategy is like the *counting on to* strategy, but it begins at the minuend rather

1 The descriptions of strategies presented here are the current descriptions used by our team, and we consider them fluid, as our understanding of these ideas continues to evolve. For a more detailed discussion of these terms, consider reading Carpenter et al. (2015).

than the subtrahend. The teachers in this group acknowledged this strategy is possible, but they observed that it is used by students infrequently. They acknowledged it as a valid strategy, and they decided to be on the lookout for it.

Fact Recall

A student using *fact recall—derived facts* uses a related, known fact to help solve a problem involving an unknown fact. For example, when evaluating 17 – 8, the student might decompose the eight into a seven and a one, subtract the seven from 17 to get ten, and then subtract the remaining one from the ten to find the difference.

Note: Students may use many other strategies that involve *derived facts* while solving number-fact problems. For more information about derived facts, read chapter three of Carpenter et al. (2015).

A *fact recall—known fact* strategy involves the student's recalling the relevant fact directly from memory. When asked how they got the answer, students often respond that they "just knew" the answer.

After the students completed the set of problems, the teachers classified the student according to the categories listed above. The teachers observed that very few students applied the same strategy on every problem, but they reported the most prevalent strategy each child used and wrote the student's name in the corresponding column. Figure 1 shows a representation of their classification of the students in the class.

The teachers noticed that many students used a *counting down* strategy and that many others used *direct modeling* to determine the solutions. The teachers therefore developed the following learning goal:

> *Students will begin using derived facts for subtraction-fact-fluency problems.*

The teachers decided that they should develop a set of problems that would help move some or all of these students toward *derived facts* strategies. As students use a *counting down* strategy, a next step for them can be to begin thinking about subtracting numbers other than one at a time to get to landmark numbers such as ten. The teachers conjectured that, because many students were *counting down*, subtracting to a landmark of ten would be an idea likely to succeed for this class.

The teachers also engaged in extended discussion of one student's *direct modeling* strategy, in which the student used what they decided to call *invisible fingers*. To solve a problem like 12 – 4, the student would hold ten fingers up and imagine two additional fingers next to the ten to make a total of 12. The student would count down "1, 2" with a nod of the head each time, then contin-

Direct modeling	Counting down	Counting on to	Fact recall—derived facts	Fact recall—known facts
Lily K.	Zaara	Justin	Miah	Keerti
Lilly C.	Kyler	Jordan	Max	Karthik
Braxton	Ty			
Marina	Georgie			
Braylon	Delaney			
	Daniella			
	Alvero			

Figure 1. Classification of the students on the basis of the strategies they most often.

ue "3, 4," putting one finger down for each count, and say the answer was eight (represented by the number of fingers remaining). The teachers saw this strategy as an opportunity to examine the structure of our base-ten number system and a possible way to encourage students to start using fact-based strategies rather than simply counting by ones.

Planning for the Lesson

Focusing on highlighting the number ten as an important landmark, these teachers worked to develop a series of expressions and equations that might help students to start using *fact recall—derived fact* strategies. The teachers decided on the following sequence:

14 - 4
13 - 3
17 - ___ = 10
12 - 2
12 - 4
13 - 6
14 - 8

The numbers selected in the first two expressions were intended to focus the students' attention on noticing that subtracting the number in the ones place of the minuend would get the student to a landmark of ten. The teachers then agreed that they should determine whether this fact had been noticed by providing an equation with an unknown subtrahend. This unknown quantity would be represented by the number students needed to subtract from 17 to make ten. They planned to provide a follow-up expression in which subtracting would provide an answer of ten. This expression would be used as a back-up if necessary to help the class see that subtracting the value of the digit in the ones place from a number in the teens always provides an answer of ten.

The teachers selected combinations of numbers in the next three problems in such a way that the students could break the subtrahend into parts, first subtract a part to get to ten, and then subtract the remaining part to determine the final solution. Number facts involving doubles (e.g., 3 + 3 = 6, 2 + 2 = 4) were selected, because the students in this classroom appeared to know their addition facts involving doubles (i.e., n + n) better than other facts. The teachers planned to provide these items one at a time and to have students share their thinking about each item. During the lesson, the teacher planned to observe student strategies as carefully as possible and to seek to identify students using ten as a landmark for subtraction. These students would then be asked to share their ideas with the other students during the lesson.

Strategy for differentiation to meet the needs of all students in the class

This lesson was developed to be a whole-group discussion that allowed the students time to work on and discuss each problem separately. One way to ensure that all learners could address these problems was to provide manipulatives and dry-erase boards (with markers) for students to use while they tried to solve the problems.

To advance the understanding of those students who were already using *fact recall—derived fact* and *fact recall—known fact* strategies, they agreed the teacher should ask these students to look for and articulate similarities and differences among the various shared strategies. In addition to noticing connections among different ways of thinking, exercising their communication skills by having them explain their strategies to their peers could help them to advance their mathematical understanding and communication ability. The teachers also planned to encourage students who solved these problems very quickly to think about other ways to solve them. This practice would keep students engaged with the content and make it less likely they would disrupt other students.

Notes on what to notice about student thinking

The main focus during the work time is for the teacher to listen and look for students who are using a *fact recall—derived facts* strategy. For ex-

The teacher first drew the fingers and had Lily pause at ten as she was counting backwards....

This demonstration appeared to help a few students "see" the 10, while others still grappled with the idea, still seeing this just as a *count down* strategy.

ample, they might look for those students who subtract back to ten and then subtract the remaining amount, saying, e.g., when solving 13 – 6, "13 – 3 = 10, and 10 – 3 = 7." This convenient and intentional decomposition may be one of the first *fact recall—derived facts* strategies students might use on this problem.

As previously mentioned, the teacher should also look for students during the lesson who use the "imaginary fingers," which can serve as an example of how to use ten as a landmark. If this strategy is observed, the teacher should work with this student to provide an example to the class in an effort to help the students see the structure and regularity surrounding the landmark of ten.

Lesson Plan

This lesson was developed on the basis of the goal set in the ***Analyzing Student Thinking*** section:

Students will begin using derived facts for subtraction-fact-fluency problems.

It was designed to follow a series of events, which would be repeated a number of times. The teacher provides the expression or equation. The teacher allows time for the students to work on the problem and then directs a class discussion about the strategies the students report using. Each cycle of four steps should take approximately six to nine minutes and can be repeated up to seven times (once for each of the following problems).

14 – 4
13 – 3
17 – ___ = 10
12 – 2
12 – 4
13 – 6
14 – 8

1. Show the first expression to the class by writing it on chart paper. Instruct the students to solve the problem in whatever way makes sense to them.
2. Provide time for the students to solve the problem and record their thinking using numbers, pictures, or words. During work on the first two problems, look for students who notice a pattern and express that, every time, the answer is ten, because subtracting the number in the ones place of a minuend in the teens always yields an answer of ten.

3. Consider looking for students who use ten as a landmark for subtraction, particularly on the last three problems. Also be aware of students who use "imaginary fingers." By highlighting this strategy, teachers can help students see how the number of imaginary fingers is related to the number that must be subtracted to reach ten. This strategy can be shared to provide students with an additional visual cue of how to use ten as a landmark.

4. On the basis of the strategies observed while students were working, invite individual students to share strategies they used that in-

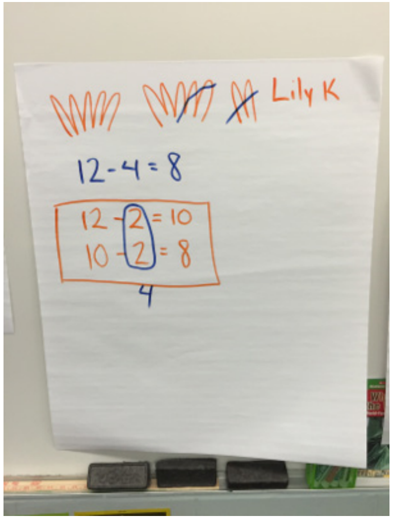

Figure 2. Lily's strategy for solving 12-4

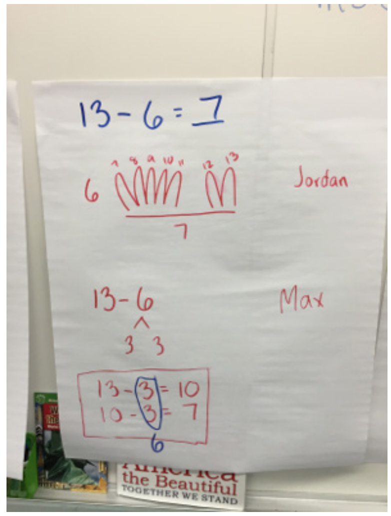

Figure 3. Jordan's and Max's strategy for solving 13-6

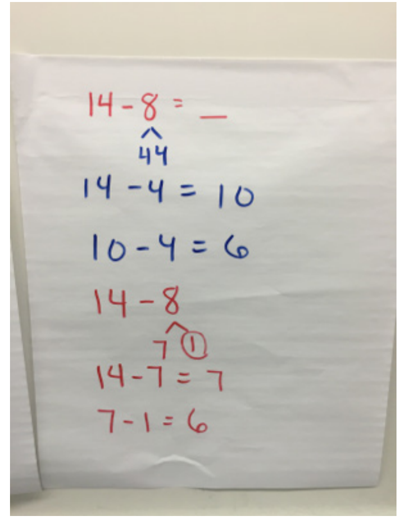

Figure 4. Two students' strategies for solving 14-8

volved subtracting to a ten. For each strategy shared, make sure to chart the students' thinking[2]. Consider sequencing the student presentations in a way that leads from less sophisticated to more sophisticated strategies and makes clear the relations among the strategies to permit comparison and contrast. For example, when discussing 12 – 4, you might have students who use the "imaginary fingers" strategy share first. Doing so may help some other students see the structure of the base-ten number system. Next could be students who use more formal notation, such as 12 – 2 = 10 and 10 – 2 = 8.

Reflection

What we learned about the students

During the lesson, as she did during the interview, Lily K. used the invisible fingers strategy so solve 12 – 4. The teacher used Lily's strategy as an example of using ten as a landmark. She asked others to restate how Lily was thinking. The students had a difficult time understanding what was being asked. In later discussion, the teachers decided that a better question to ask of students would be, "How would Lily's invisible fingers strategy work on 13 – 5?"

Figures 2–4 were created by the teacher during the lesson as a means of displaying what students described as their thinking progressed in a written format for the other students to see. A brief explanation of the strategy follows each of the figures.

The teacher first chose to share a *count down* strategy for 12 – 4 where the student,.(Lily K., Figure 2), used the *invisible fingers* strategy previously discussed. The teacher worked diligently at getting the class to see the structure of ten in the student's *invisible fingers* strategy. The teacher first drew the fingers and had Lily pause at ten as she was counting backwards. The teacher then recorded this strategy as 12 – 2 = 10 and 10 – 2 = 8 (seen in orange below the fingers). This demonstration appeared to help a few students "see" the 10, while others still grappled with the idea, still seeing this just as a *count down* strategy.

Two students' strategies for 13 – 6 were shared. The first, Jordan (Figure 3), counted up using his fingers from six to 13. His answer could be seen in the number of fingers he was holding up after his count from six to 13. The second student, Max (Figure 3), decomposed the six into two threes. He then solved 13 – 3 to get to ten, then 10 – 3 to get to 7.

2 *See the **Reflection** section for examples of how to record individual student responses.*

Two students' strategies for solving 14 – 8 were also discussed. The first (shown in blue in Figure 4) shows how a student used a *fact recall—derived facts* strategy to subtract to reach ten. The student first decomposed the eight into two fours, then subtracted four from 14 to get to ten, then subtracted the other four from ten to get to six. The second strategy (shown in red in Figure 4) relies on the knowledge of a doubles fact. The student first decomposed the eight into a seven and a one, then used the known fact of 14 – 7 = 7. From there, the student subtracted the one from the seven to get six.

Reflections on the lesson

Posting the representations of solutions to the problems for all the students to see allowed them to engage with the problems and provided a good opportunity for the teacher to gain insight into how individual students thought about them.

In the postlesson follow up, the teachers revisited the decision to make the final three problems similar in their structure—that is, making each of the subtrahends exactly twice the number in the ones place in the minuend. The intent of that structure was to help students leverage their existing understanding of doubles. For example, in 14 – 8, the eight could be decomposed into two fours, subtraction of one of which would yield ten.

This idea seemed to work as intended during the implementation of the lesson; some of the students seemed to come to understand these strategies as their peers explained them. Not all of the students did, though. In the postlesson discussion, however, the teachers agreed that replacing the final 14 – 8 with a problem in which the subtrahend could *not* be decomposed into equal two parts—either of which could be subtracted from the minuend to yield ten—would be an improvement to the lesson. Doing so would prevent students from concluding that that this strategy only works on problems where the subtrahend is exactly twice the number in the ones place in the minuend. They identified 13 – 7 and 14 – 6 as good options.

What's Next?

Stories of teachers engaging in collaborative inquiry focused on using student thinking to inform instructional decisions

What's Next? is a collection of stories documenting professional development experiences shared by elementary teachers working collaboratively to study the complex process of teaching and learning mathematics. Each story in the collection describes practicing teachers studying the thinking processes of real students and using what they learn about those students to make decisions and try to help advance those students' understanding on that day.

The teachers in the stories start by learning about how individual students solve a set of mathematics problems. They use this freshly gathered knowledge of student thinking to develop near-term learning goals for students and a lesson plan tailored to specific students on that specific day. One of the teachers implements the planned lesson while the others observe in real time. The teachers then gather to discuss and reflect on their observations and insights.

In these lessons, the practice of teaching is slowed way down. The stories tell of teachers who are studying student thinking and using that information to plan and implement instructional decisions at a pace that is much slower than is typical in daily practice. They depict a process with many aspects in common with formative assessment and lesson study, both of which are also conceptualized as processes and not outcomes.

The stories depict real situations that occurred in real time and include both successes and shortcomings. We hope they will be studied and discussed by interested educators so that the lessons and ideas experienced by these teachers and instructional coaches will contribute to additional learning and sharing among other interested teachers.

Learn more about these and other stories at http://www.teachingisproblemsolving.org/